Coupled higher-order nonlinear Schrödinger equations: a new integrable case via the singularity analysis

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Abstract

A new system of coupled higher-order nonlinear Schrödinger equations is proposed which passes the Painlevé test for integrability well. A Lax pair and a multi-field generalization are obtained for the new system.

1 Introduction

In this paper, we study the integrability of the following system of two symmetrically coupled higher-order nonlinear Schrödinger equations:

$$\begin{gathered} q_t = hq_{xxx} + aq\bar{q}q_x + bq^2\bar{q}_x + cr\bar{r}q_x + dq\bar{r}r_x + eqr\bar{r}_x + \\ \mathrm{i}(sq_{xx} + fq^2\bar{q} + gqr\bar{r}), \\ r_t = hr_{xxx} + ar\bar{r}r_x + br^2\bar{r}_x + cq\bar{q}r_x + dr\bar{q}q_x + erq\bar{q}_x + \\ \mathrm{i}(sr_{xx} + fr^2\bar{r} + grq\bar{q}), \end{gathered} \tag{1}$$

where $h, a, b, c, d, e, s, f, g$ are real parameters, $h \neq 0$, and the bar denotes the complex conjugation.

Section 2 is devoted to the singularity analysis of (1), and there, under certain simplifying assumptions, we find the following new case when the system (1) passes the Painlevé test for integrability well:

$$a \neq 0, \quad b = 0, \quad c = d = e = a, \quad f = g = \frac{as}{3h}. \tag{2}$$

In Section 3, we prove the integrability of (1) with (2), constructing a corresponding 4×4 Lax pair, and then obtain a multi-field generalization of the new integrable system.

Section 4 contains some concluding remarks.

2 Singularity analysis

Let us apply the Weiss-Kruskal algorithm of the singularity analysis [1], [2] to the system (1) (we set $h = 1$ w.l.g.). With respect to $q, \bar{q}, r, \bar{r}$, which should be considered as mutually independent during the Painlevé test, the system (1) is a normal system of four third-order equations, of total order twelve. A hypersurface $\phi(x,t) = 0$ is non-characteristic for (1) if $\phi_x \neq 0$, and we set $\phi_x = 1$. Then we substitute the expansions

$$\begin{aligned} q &= q_0(t)\phi^{\alpha} + \ldots + q_n(t)\phi^{n+\alpha} + \ldots, \\ \bar{q} &= \bar{q}_0(t)\phi^{\beta} + \ldots + \bar{q}_n(t)\phi^{n+\beta} + \ldots, \\ r &= r_0(t)\phi^{\gamma} + \ldots + r_n(t)\phi^{n+\gamma} + \ldots, \\ \bar{r} &= \bar{r}_0(t)\phi^{\delta} + \ldots + \bar{r}_n(t)\phi^{n+\delta} + \ldots \end{aligned} \tag{3}$$

(the bar does not mean the complex conjugation now) into (1) and obtain four algebraic equations for $\alpha, \beta, \gamma, \delta, q_0\bar{q}_0, r_0\bar{r}_0$, which determine the dominant behavior of solutions near $\phi = 0$, as well as one twelfth-degree algebraic equation with respect to n, which determines the positions of resonances in the expansions. The perfect analysis of those five equations is very complicated, and we will publish it later on. But now we impose certain restrictions on the expansions (3) in order to reach a new integrable case of (1) by a short way.

For this purpose, we set $\alpha = \beta = \gamma = \delta = -1$ and require that exactly two of twelve resonances lie in the position $n = 0$. Under these simplifying assumptions, we find from (1) and (3) that $q_0\bar{q}_0 = r_0\bar{r}_0 = \text{constant} \neq 0$ (we set constant $= 1$ w.l.g.), and that

$$a = -6 - b - c - d - e, \tag{4}$$

$$\begin{gathered} (n+1)n^2(n-3)(n-4)\times \\ (n^2 - 6n - 2b - 2d + 5)(n^2 - 6n - 2b - 2e + 5)\times \\ (n^3 - 6n^2 + (5 - 2d - 2e)n + 4(c + d + e + 3)) = 0. \end{gathered} \tag{5}$$

case #	n_1	n_3	n_5	n_6	all resonances	n_{log}
1	1	1	1	1	$-1,0,0,1,1,1,1,3,4,4,5,5$	1
2	1	1	1	2	$-1,0,0,1,1,1,2,3,3,4,5,5$	2
3	1	1	2	2	$-1,0,0,1,1,2,2,2,3,4,5,5$	2
4	1	2	1	1	$-1,0,0,1,1,1,2,3,4,4,4,5$	1
5	1	2	1	2	$-1,0,0,1,1,2,2,3,3,4,4,5$	2
6	1	2	2	2	$-1,0,0,1,2,2,2,2,3,4,4,5$	2
7	1	3	1	1	$-1,0,0,1,1,1,3,3,3,4,4,5$	1
8	1	3	1	2	$-1,0,0,1,1,2,3,3,3,3,4,5$	3
9	1	3	2	2	$-1,0,0,1,2,2,2,3,3,3,4,5$	2
10	2	1	1	1	$-1,0,0,1,1,1,2,3,4,4,4,5$	1
11	2	1	1	2	$-1,0,0,1,1,2,2,3,3,4,4,5$	2
12	2	1	2	2	$-1,0,0,1,2,2,2,2,3,4,4,5$	2
13	2	2	1	1	$-1,0,0,1,1,2,2,3,4,4,4,4$	1
14	2	2	1	2	$-1,0,0,1,2,2,2,3,3,4,4,4$	NO, if (8)
15	2	3	1	1	$-1,0,0,1,1,2,3,3,3,4,4,4$	1
16	2	3	1	2	$-1,0,0,1,2,2,3,3,3,3,4,4$	3
17	2	3	2	2	$-1,0,0,2,2,2,2,3,3,3,4,4$	2
18	3	1	1	1	$-1,0,0,1,1,1,3,3,3,4,4,5$	1
19	3	1	1	2	$-1,0,0,1,1,2,3,3,3,3,4,5$	3
20	3	1	2	2	$-1,0,0,1,2,2,2,3,3,3,4,5$	2
21	3	2	1	1	$-1,0,0,1,1,2,3,3,3,4,4,4$	1
22	3	2	1	2	$-1,0,0,1,2,2,3,3,3,3,4,4$	3
23	3	2	2	2	$-1,0,0,2,2,2,2,3,3,3,4,4$	2

Table 1: Positions of resonances and logarithmic terms.

Due to (5), five resonances lie in the positions $n=-1,0,0,3,4$. Denoting the positions of other seven resonances as $n_1,n_2,\ldots,n_7$, we find from (5) that

$$\begin{gathered}
n_2=6-n_1,\quad n_4=6-n_3,\quad n_7=6-n_5-n_6,\\
d=\tfrac{1}{2}(5-2b-6n_1+n_1^2),\quad e=\tfrac{1}{2}(5-2b-6n_3+n_3^2),\\
b=\tfrac{1}{4}(5-6n_1+n_1^2-6n_3+n_3^2+6n_5-n_5^2+6n_6-n_6^2-n_5n_6),\\
c=\tfrac{1}{4}(-22+12n_5-2n_5^2+12n_6-2n_6^2-8n_5n_6+n_5^2n_6+n_5n_6^2).
\end{gathered}\tag{6}$$

We require that the considered branch is generic, i.e. eleven resonances lie in nonnegative positions. Taking into account the admissible multiplicity of resonances, we have to study 23 distinct cases listed in Table 1.

At the next step of the analysis, we find from (1) and (3) the recursion relations for $q_n,\bar{q}_n,r_n,\bar{r}_n,\ n=0,1,2,\ldots,$ and then check the consistency

of those relations at the resonances, using the *Mathematica* system [3] for computations. The results are listed in Table 1, in its column n_{log}, where n_{log} denotes the position in which some logarithmic terms should be introduced into the expansions (3) due to the following reasons: either the actual number of arbitrary functions is less than the multiplicity of the resonance, or the compatibility conditions at the resonance cannot be satisfied identically. As we see, all the cases but one, #14, have already failed to pass the Painlevé test. In the case #14, where

$$h = 1, \quad a = c = d = e = -\frac{3}{2}, \quad b = 0 \tag{7}$$

due to our assumption and formulae (4) and (6), we have to set

$$f = g = -\frac{1}{2}s \tag{8}$$

for the compatibility conditions at $n = 2, 3$ to become identities.

The system (1) with (7) and (8) admits many branches, i.e. kinds of expansions (3). We have already studied the generic branch. Other branches either are Taylor expansions governed by the Cauchy-Kovalevskaya theorem, or are related to the following two:

1. $\alpha = -1$, $\beta = -1$, $\gamma = -2$, $\delta = 2$, $q_0\bar{q}_0 = 4$, $\forall\, r_0, \bar{r}_0$, positions of resonances are $n = -4, -1, 0, 0, 0, 1, 1, 3, 4, 4, 5, 5$;

2. $\alpha = -2$, $\beta = 0$, $\gamma = -3$, $\delta = 2$, $q_0\bar{q}_0 = 8$, $\forall\, r_0, \bar{r}_0$, positions of resonances are $n = -5, -2, -1, 0, 0, 0, 2, 4, 5, 5, 6, 7$.

Compatibility conditions at all resonances of these branches turn out to be identities. The Painlevé test is completed.

Since we consider q and $\bar{q}$, r and $\bar{r}$ as mutually independent, evident scale transformations of $q, \bar{q}, r, \bar{r}$ and t relate the conditions (7) and (8) with the more general condition (2). On the other hand, the transformation

$$\begin{gathered} q' = \tfrac{1}{2} q \exp\omega, \quad \bar{q}' = -\tfrac{1}{2}\bar{q}\exp(-\omega), \\ r' = \tfrac{1}{2} r \exp\omega, \quad \bar{r}' = -\tfrac{1}{2}\bar{r}\exp(-\omega), \\ \omega = \tfrac{\mathrm{i}s}{3}x + \tfrac{2\mathrm{i}s^3}{27}t, \ x' = x + \tfrac{s^2}{3}t, \ t' = -t \end{gathered} \tag{9}$$

changes the system (1) with (7) and (8) into the system of four coupled mKdV equations

$$\begin{gathered} q_t + q_{xxx} + 6q\bar{q}q_x + 6(qr\bar{r})_x = 0, \\ \bar{q}_t + \bar{q}_{xxx} + 6q\bar{q}\bar{q}_x + 6(\bar{q}r\bar{r})_x = 0, \\ r_t + r_{xxx} + 6r\bar{r}r_x + 6(q\bar{q}r)_x = 0, \\ \bar{r}_t + \bar{r}_{xxx} + 6r\bar{r}\bar{r}_x + 6(q\bar{q}\bar{r})_x = 0, \end{gathered} \tag{10}$$

where the prime of $x, t, q, \bar{q}, r, \bar{r}$ is omitted, and the bar does not mean (but may mean) the complex conjugation. This form (10) is useful for obtaining a Lax pair for the new integrable case (2) of (1).

3 Lax pair and generalization

Let us consider the linear problem

$$\Psi_x = U\Psi, \quad \Psi_t = V\Psi \tag{11}$$

with the matrices U and V given in the following block form [4]:

$$U = \mathrm{i}\zeta \begin{pmatrix} -I_1 & 0 \\ 0 & I_2 \end{pmatrix} + \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}, \tag{12}$$

$$\begin{gathered} V = \mathrm{i}\zeta^3 \begin{pmatrix} -4I_1 & 0 \\ 0 & 4I_2 \end{pmatrix} + \zeta^2 \begin{pmatrix} 0 & 4Q \\ 4R & 0 \end{pmatrix} + \\ \mathrm{i}\zeta \begin{pmatrix} -2QR & 2Q_x \\ -2R_x & 2RQ \end{pmatrix} + \begin{pmatrix} Q_xR - QR_x & -Q_{xx} + 2QRQ \\ -R_{xx} + 2RQR & R_xQ - RQ_x \end{pmatrix}, \end{gathered} \tag{13}$$

where I_1 and I_2 are unit matrices, ζ is a parameter. The compatibility condition of the linear problem (11),

$$U_t - V_x + UV - VU = 0, \tag{14}$$

becomes the system of two matrix mKdV equations [5]:

$$\begin{aligned} Q_t + Q_{xxx} - 3Q_xRQ - 3QRQ_x = 0, \\ R_t + R_{xxx} - 3R_xQR - 3RQR_x = 0. \end{aligned} \tag{15}$$

If we substitute

$$Q = \begin{pmatrix} q & r \\ \bar{r} & \bar{q} \end{pmatrix}, \quad R = -\begin{pmatrix} \bar{q} & r \\ \bar{r} & q \end{pmatrix} \tag{16}$$

into (15), we obtain exactly the new system (10). This proves that the new case (2) of the coupled higher-order nonlinear Schrödinger equations (1) possesses a parametric Lax pair.

A multi-field generalization of the system (10) is obtained by choosing

$$\begin{aligned} Q &= \begin{pmatrix} u_0 I\otimes I + \sum_{k=1}^{2m-1} u_k e_k \otimes I & v_0 I\otimes I + \sum_{k=1}^{2m-1} v_k I\otimes e_k \\ v_0 I\otimes I - \sum_{k=1}^{2m-1} v_k I\otimes e_k & u_0 I\otimes I - \sum_{k=1}^{2m-1} u_k e_k \otimes I \end{pmatrix}, \\ R &= -\begin{pmatrix} u_0 I\otimes I - \sum_{k=1}^{2m-1} u_k e_k \otimes I & v_0 I\otimes I + \sum_{k=1}^{2m-1} v_k I\otimes e_k \\ v_0 I\otimes I - \sum_{k=1}^{2m-1} v_k I\otimes e_k & u_0 I\otimes I + \sum_{k=1}^{2m-1} u_k e_k \otimes I \end{pmatrix}, \end{aligned} \tag{17}$$

where I is the $2^{m-1} \times 2^{m-1}$ unit matrix, and $\{e_1, \dots, e_{2m-1}\}$ are $2^{m-1} \times 2^{m-1}$ anti-commutative and anti-Hermitian matrices:

$$\{e_i, e_j\}_+ = -2\delta_{ij} I, \quad e_k^\dagger = -e_k. \tag{18}$$

Then the compatibility condition (14) becomes the system

$$\begin{aligned} &u_{j,t} + u_{j,xxx} + 6 \textstyle\sum_{k=0}^{2m-1} u_k^2 u_{j,x} + 6 \left(\sum_{k=0}^{2m-1} v_k^2 u_j \right)_x = 0, \\ &v_{j,t} + v_{j,xxx} + 6 \textstyle\sum_{k=0}^{2m-1} v_k^2 v_{j,x} + 6 \left(\sum_{k=0}^{2m-1} u_k^2 v_j \right)_x = 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 0, 1, \dots, 2m-1. \end{aligned} \tag{19}$$

If we assume that u_k and v_k are real and set

$$\begin{aligned} u_{2j-2} + \mathrm{i} u_{2j-1} = q_j, \\ v_{2j-2} + \mathrm{i} v_{2j-1} = r_j, \end{aligned} \quad j = 1, 2, \dots, m, \tag{20}$$

the system (19) is expressed as

$$\begin{aligned} &q_{j,t} + q_{j,xxx} + 6 \textstyle\sum_{k=1}^{m} |q_k|^2 q_{j,x} + 6 \left(\sum_{k=1}^{m} |r_k|^2 q_j \right)_x = 0, \\ &r_{j,t} + r_{j,xxx} + 6 \textstyle\sum_{k=1}^{m} |r_k|^2 r_{j,x} + 6 \left(\sum_{k=1}^{m} |q_k|^2 r_j \right)_x = 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 1, 2, \dots, m. \end{aligned} \tag{21}$$

4 Conclusion

In the literature, the following three integrable cases of coupled higher-order nonlinear Schrödinger equations (1) are known and studied (sometimes in a form of coupled mKdV equations):

$$a \neq 0, \quad b = e = 0, \quad c = d = \frac{1}{2} a, \quad f = g = \frac{as}{3h}, \tag{22}$$

$$a \neq 0, \quad b = d = e = 0, \quad c = a, \quad f = g = \frac{as}{3h}, \tag{23}$$

$$a \neq 0, \quad b = d = e = \frac{1}{3} a, \quad c = \frac{2}{3} a, \quad f = g = \frac{2as}{9h}; \tag{24}$$

they were introduced in [6], [7], [8], respectively.

The new integrable case (2) of the system (1), obtained in this paper by means of the singularity analysis, admits the reduction $r \to 0$ to the Hirota equation [9] and the reduction $r \to q$ to the Sasa-Satsuma equation [10]. Its soliton solutions and conservation laws deserve further investigation.

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